Attractors and Subshifts of Finite Type of ECA 41

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Abstract— *In this paper, the dynamics of elementary cellular automaton rule 41 is investigated in the bi-infinite symbolic sequence space. In spite of rule 41 is not surjective, but it possess of rich and complex dynamical behaviors. The existence of attractors and subshifts of finite type of the rule's global map is strictly proved, some interesting dynamical properties on these subshifts, such as positive topological entropies, topological transitivity and topological mixing, chaos in the sense of Li-Yorke and Devaney, are revealed.*

Keywords: attractor; cellular automata; chaos; subshift of finite type; symbolic dynamics.

1. Introduction

Cellular automata (CA) are a class of spatially and temporally discrete, deterministic mathematical systems characterized by local interactions and an inherently parallel form of evolution. CA, formally introduced by von Neumann and Ulam in the 1940's to the 1950's, are able to produce complex dynamical phenomena by means of designing simple local rules [1]. Due to their simple mathematical constructions and distinguishing features, CA have been widely used to model a variety of dynamical systems. The study of topological dynamics of CA began with Hedlund in 1969, who viewed one-dimensional CA (1D CA) in the context of symbolic dynamics as endomorphisms of the shift dynamical system [2], where the main results are the characterizations of surjective and open CA. Based on the theoretical concept of universality, researchers have tried to develop even simpler and more practical architectures of CA which can be used for widely diverse applications. In the early 1980's, Wolfram introduced space-time representations of 1D CA and informally classified them into four classes by using dynamical concepts like periodicity, stability and chaos [3, 4]. In 2002, he introduced his monumental work *A New Kind of Science* [5]. To provide a rigorous foundation for Wolfram's empirical observations Chua *et al* derived a nonlinear dynamics perspective to elementary cellular automata (ECA) via the concepts like characteristic function, forward time-τ map, basin tree diagram and Isle-of-Eden digraph [6-8]. It was known that there are 256 ECA rules, only 88 rules are globally independent from each other [9]. These 88 global independent ECA rules are also organized into 4 groups with distinct qualitative dynamics: 40 period-k ($k = 1, 2, 3, 6$) rule classes, 30 topologically distinct Bernoulli shift rule classes, 10 complex Bernoulli shift rule classes and 8 hyper Bernoulli shift ones [6-9].

CA are dynamical systems with a very rich spectrum of dynamical properties. Although topological properties of CA can be explored, many of them such as topological entropy, sensitivity, topological mixing, topological transitivity and so on are undecidable. The relationship between positively expansive and mixing was investigated by Blanchard and Maass [10]. The transitive CA implies surjective and sensitive to initial conditions have been obtained by Margara and Kurka [11, 12]. The dynamics of a specific ECA on their Bernoulli-shift invariant subset was analysed [13-15]. When a cellular automaton is not surjective, the concept of an attractor is essential for its understanding. Some attractors of CA are subshifts and some are not. These two kinds of attractor have quite different properties [16].

ECA rule 41, which is not a surjective CA, possess of rich and complex dynamical behaviors. In this paper, the dynamics of the rule's global map is investigated in the bi-infinite symbolic sequence space. The existence of attractors and subshifts of finite type of the rule's global map is strictly proved, some dynamical properties on these subshifts are revealed. As an illustration, we give a simulation of the evolution of rule 41 with a random initial configuration in Figure 1, where the black pixel stands for 1 and white for 0.

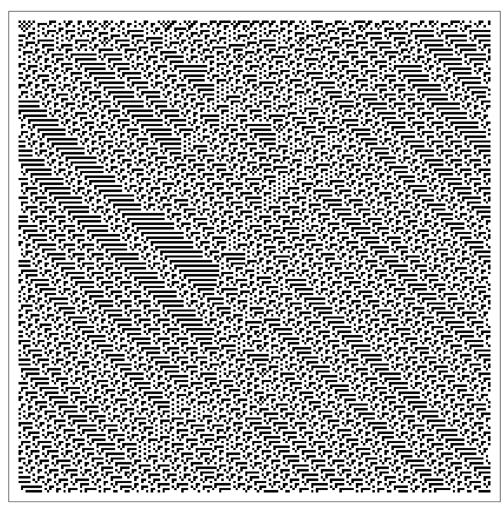

Fig. 1: The evolution of rule 41, from a random initial configuration.

The rest of this paper is organized as follows: Section 2 presents the preliminaries of symbolic dynamical systems and CA, and two lemmas. Section 3 explores two attractors and some subshifts of finite type of rule's global map f_{41}. Section 4 demonstrates some complex dynamics of f_{41}. Finally, Section 5 concludes this paper.

2. Preliminaries

The bi-infinite binary symbols sequence space is a configuration set on $S = \{0, 1\}$:

$$\Sigma_2 = \{x = (\cdots, x_{-1}, \overset{*}{x}_0, x_1, \cdots) \mid x_i \in S, i \in Z\}$$

and the metric "d" on Σ_2 defined as

$$d(x, y) = \max_{i \in Z}\{\frac{\rho(x_i, y_i)}{2^{|i|}}\}$$

for any x, $y \in \Sigma_2$, where $\rho(\cdot, \cdot)$ is the metric on S defined as

$$\rho(x_i, y_i) = \begin{cases} 0, & if \quad x_i = y_i \\ 1, & if \quad x_i \neq y_i. \end{cases}$$

It is known that Σ_2 is a compact, perfect and totally disconnected metric space.

A finite sequence $a = (a_0, a_1, \cdots, a_{n-1})$ $(a_i \in S,\ i = 0, 1, 2, \cdots, n-1)$ is called a word over S. If $x \in \Sigma_2$ and $I = [i, j]$ is an interval of integers, put $x_{[i,j]} = (x_i, x_{i+1}, \cdots, x_j)$ $(i < j)$, $x_{[i,j)} = (x_i, \cdots, x_{j-1})$. For a word $a = (a_0, \cdots, a_{n-1})$, if there exists an $n \in Z$ such that $x_{n+k} = a_k$ $(k = 0, 1, \cdots, n-1)$, then say a be a subword of x, denoted by $a \prec x$; otherwise $a \not\prec x$.

The left-shift σ_L and right-shift σ_R are defined by

$$\sigma_L(\cdots, x_{-1}, \overset{*}{x}_0, x_1, \cdots) = (\cdots, x_0, \overset{*}{x}_1, x_2 \cdots)$$

and

$$\sigma_R(\cdots, x_{-1}, \overset{*}{x}_0, x_1, \cdots) = (\cdots, x_{-2}, \overset{*}{x}_{-1}, x_0, \cdots)$$

respectively.

By a theorem of Hedlund [3], a map $f : \Sigma_2 \to \Sigma_2$ is a cellular automaton iff it is continuous and commutes with σ, i.e. $\sigma \circ f = f \circ \sigma$, where σ is left-shift or right-shift.

A set $X \subseteq \Sigma_2$ is f-invariant if $f(X) \subseteq X$, and strongly f-invariant if $f(X) = X$. If X is a closed and f-invariant, then (X, f) or simply X is called a subsystem of f.

The omega-limit of a closed invariant set X is $\Omega_f(X) = \bigcap_{n \geq 0} f^n(X)$. A set $X \subset \Sigma_2$ is an attractor of f if there exists a clopen f-invariant set Y such that $\Omega_f(Y) = X$. The maximal attractor $\Omega_f = \Omega_f(\Sigma_2)$ is also called the limit set of f. A subshift is a non-empty subset $\Lambda \subset \Sigma_2$ which is strongly σ-invariant and closed. A subshift attractor of a cellular automaton is an attractor which is a subshift. For example, the maximal attractor is a subshift attractor [19].

Definition *1:* A closed and open (clopen) f-invariant set $U \subset \Sigma_2$ is spreading if $f^k(U) \subset \sigma_L(U) \cap U \cap \sigma_R(U)$ for some $k > 0$.

Lemma 1: [16] Let $f : \Sigma_2 \to \Sigma_2$ be a cellular automaton and U a clopen f-invariant set, then $\Omega_f(U)$ is a subshift attractor iff U is spreading.

Table 1: Truth table of Boolean function of Rule 41

(x_{i-1}, x_i, x_{i+1})	$\hat{f}(x_{i-1}, x_i, x_{i+1})$
(0, 0, 0)	1
(0, 0, 1)	0
(0, 1, 0)	0
(0, 1, 1)	1
(1, 0, 0)	0
(1, 0, 1)	1
(1, 1, 0)	0
(1, 1, 1)	0

For any subshift Λ, there exists a set $\mathscr{A}$ consisting of some words over $S = \{0, 1\}$ such that $\Lambda = \Lambda_{\mathscr{A}} = \{x \in \Sigma_2 | \ a \not\prec x, \forall\ a \in \mathscr{A}\}$, the set $\mathscr{A}$ named excluded block system. A subshift is of finite type if the set $\mathscr{A}$ is finite. The order of the finite type subshift denoted by N, which is the length of the longest word in $\mathscr{A}$.

Lemma 2: [17] For any subshift of finite type Λ, the followings statements are equivalent:
(1) there exists a set $\mathscr{A}$ consisting of some words over S such that $\Lambda = \Lambda_{\mathscr{A}} = \{x \in \Sigma_2 | \ a \not\prec x, \forall\ a \in \mathscr{A}\}$;
(2) there exists a set $\mathscr{B}$ consisting of some words over S such that $\Lambda = \{x \in \Sigma_2 | \ x_{[n,n+N-1]} \in \mathscr{B}, \forall n \in Z\}$, where N is the order of Λ. The set $\mathscr{B}$ named determinative block system of Λ.

It is well known that each ECA rule can be expressed by a Boolean function. For example, the one of rule 41 is a local map $\hat{f}$:

$$\hat{f}(x_{i-1}, x_i, x_{i+1}) \\ = x_{i-1} \cdot \overline{x}_i \cdot x_{i+1} \oplus \overline{x}_{i-1} \cdot x_i \cdot x_{i+1} \oplus \overline{x}_{i-1} \cdot \overline{x}_i \cdot \overline{x}_{i+1}$$

where "$\cdot$", "$\oplus$", and "$\_$" stand for "AND", "XOR" and "NOT" logical operations respectively [7, 13]. The truth table of its Boolean function is shown in Table 1.

It is clear that its binary output sequence is 10010100. Thus, a global map $f_{41} : \Sigma_2 \to \Sigma_2$ with

$$f_{41}(\cdots, x_{-1}, \overset{*}{x}_0, x_1, \cdots) = (\cdots, y_{-1}, \overset{*}{y}_0, y_1, \cdots)$$

can be induced by $\hat{f}$, where $y_i = \hat{f}(x_{i-1}, x_i, x_{i+1})$.

The n $(n \geq 2)$ times iteration of $\hat{f}$ is a map $\hat{f}^n$ from $\{0, 1\}^{2n+1}$ to $\{0, 1\}$ with

$$\hat{f}^n(a_{-n}, \cdots, a_0, \cdots, a_n) = \\ \hat{f}(\hat{f}^{n-1}(a_{[-n,n-2]}), \hat{f}^{n-1}(a_{[-n+1,n-1]}), \hat{f}^{n-1}(a_{[-n+2,n]})).$$

3. Attractors and Subshifts of Finite Type

In this section, two attractors and some subshifts of finite type of the dynamical system (Σ_2, f_{41}) induced by rule 41 are revealed.

3.1 Attractors

Proposition 1: For rule 41, there exists an invariant subset $\Lambda \subset \Sigma_2$, such that $f_{41}(x) \in \Lambda, \forall\ x \in \Sigma_2$, where $\Lambda = \{x \in \Sigma_2 |\ a \not\prec x, \forall\ a \in \mathscr{A}_1\}$, and $\mathscr{A}_1 = \{(1,0,1,1,1), (1,1,1,0^{3k+1},1,1), (1,1,1,0^{3k+2},1,1,1),$ $k \in Z^+\}$, where 0^n is 0-constant block of length n, $n = 3k+1,\ 3k+2$.

Proof: Let $y = f_{41}(x)$, if $y_{[i,i+n-1]}$ is a word of y, then there exists a word $x_{[i-1,i+n]}$ of x, which is a pre-image of $y_{[i,i+n-1]}$, such that $\hat{f}_{41}(x_{[i-1,i+n]}) = y_{[i,i+n-1]}$.

Assume that $y_{[i,i+4]} = (1,0,1,1,1) \in \mathscr{A}_1$, and its a pre-image is $x_{[i-1,i+5]}$. It is easy to know that the pre-image set of $y_{[i,i+3]} = (1,0,1,1)$ is $\{(0,1,1,0,1,1), (1,0,1,0,1,1)\}$, and the pre-image set of $y_{[i+1,i+4]} = (0,1,1,1)$ is $\{(1,0,0,0,0,0)\}$. Since $y = f_{41}(x)$, so the pre-image of y must satisfy $x_{[i-1,i+4]} = \{(0,1,1,0,1,1)\}$ or $x_{[i-1,i+4]} = (1,0,1,0,1,1)$, and $x_{[i,i+5]} = (1,0,0,0,0,0)$. This lead to a contradiction, so the pre-image of $y_{[i,i+4]} = (1,0,1,1,1)$ is empty. Similarly, these words $(1,1,1,0^{3k+1},1,1)$ and $(1,1,1,0^{3k+2},1,1,1)$ must have no pre-image ($n = 3k+1, 3k+1, k \in Z^+$). This implies that $f_{41}(\Sigma_2) \subset \Lambda$ and $f_{41}(\Lambda) \subset \Lambda$. ∎

Theorem 1: $\Omega_f(\Lambda)$ is a subshift attractor of f_{41}, where Λ is the invariant set obtained in Proposition 1.

Proof: In fact, Σ_2 is σ_L-invariant and σ_R-invariant, and Σ_2 is a clopen set, so Σ_2 is spreading, thus, $\Omega_f(\Sigma_2) = \bigcap_{n\geq 0} f_{41}^n(\Sigma_2)$ is a subshift attractor of f_{41}. Λ obtained in Proposition 1 must satisfy $f_{41}(\Sigma_2) \subset \Lambda$ and $f_{41}(\Lambda) \subset \Lambda$. This implies that $\Omega_f(\Sigma_2) = \Omega_f(\Lambda)$, thus $\Omega_f(\Lambda)$ is a subshift attractor of f_{41}. ∎

Proposition 2: For rule 41, there exists an invariant subset $\Lambda_0 \subset \Lambda$, such that $\Omega_f(\Lambda_0) = \{0^*, 1^*\}$, where $\Lambda_0 = \{x \in \Sigma_2 |\ x_{[i,i+5]} \in \mathscr{A}_0\}$, and $\mathscr{A}_0$={(0, 0, 1, 0, 0, 1), (0, 1, 0, 0, 1, 0), (1, 0, 0, 1, 0, 0), (1, 1, 0, 0, 1, 0), (1, 1, 1, 0, 0, 1), (1, 1, 1, 1, 0, 0), (1, 1, 1, 1, 1, 0), (1, 1, 1, 1, 1, 1)}, 0^* and 1^* are the cycle configurations $0^* = (0^\infty)$ and $1^* = (1^\infty)$.

Proof: The result can be directly validated. ∎

Theorem 2: $\{0^*, 1^*\}$ is a local attractor of f_{41}.

Proof: It is easily verified that $f_{41}(x) \in \{0^*, 1^*\}$, for any $x \in \Lambda_0$. ∎

3.2 Subshifs of finite type

In this subsection, some subshifts of finite type are given out. Based on a computer-aided method, the following propositions can be easily verified:

Proposition 3: For rule 41, there exists a subset $\Lambda_1 \subset \Sigma_2$, such that $f_{41}|_{\Lambda_1} = \sigma_L|_{\Lambda_1}$. where $\Lambda_1 = \Lambda_{\mathscr{B}_1} = \{x \in \Sigma_2 |\ x_{[i-1,i+1]} \in \mathscr{B}_1, \forall\ i \in Z\}$ and the determinative block system $\mathscr{B}_1$ is a 3-sequence set, whose binary code set is $\mathscr{B}_1 = \{(0,1,0), (0,1,1), (1,0,1), (1,1,0)\}$.

For convenience, $\mathscr{B}_1$ can also be denoted by its decimal code set $D(\mathscr{B}_1) = \{2,3,5,6\}$.

Proposition 4: For rule 41, there exists a subset $\Lambda_2 \subset \Sigma_2$, such that $f_{41}^4|_{\Lambda_2} = \sigma_L^3|_{\Lambda_2}$. where $\Lambda_2 = \Lambda_{\mathscr{B}_2} = \{x \in \Sigma_2 |\ x_{[i-4,i+4]} \in \mathscr{B}_2, \forall\ i \in Z\}$ and the determinative block system $\mathscr{B}_2$ is a 9-sequence set, whose decimal code set is $D(\mathscr{B}_2)$={4, 8, 14, 15, 16, 20, 29, 30, 32, 40, 41, 58, 60, 64, 65, 76, 80, 82, 100, 116, 120, 129, 131, 133, 144, 147, 153, 161, 164, 200, 201, 232, 233, 241, 258, 263, 266, 285, 286, 288, 294, 306, 322, 328, 329, 398, 399, 400, 403, 417, 420, 455, 464, 466, 483}.

Proposition 5: For rule 41, there exists a subset $\Lambda_3 \subset \Sigma_2$,such that $f_{41}^4|_{\Lambda_3} = \sigma_R^4|_{\Lambda_3}$, where $\Lambda_3 = \Lambda_{\mathscr{B}_3} = \{x \in \Sigma_2 |\ x_{[i-4,i+4]} \in \mathscr{B}_3, \forall\ i \in Z\}$ and the determinative block system $\mathscr{B}_3$ is a 9-sequence set, whose decimal code set is $D(\mathscr{B}_3)$={0, 1, 2, 4, 5, 8, 10, 16, 20, 21, 32, 33, 34, 40, 42, 50, 57, 60, 62, 63, 64, 65, 66, 68, 76, 78, 79, 80, 85, 100, 101, 114, 120, 121, 124, 126, 127, 128, 129, 130, 133, 136, 153, 156, 158, 159, 160, 161, 170, 182, 201, 202, 214, 218, 228, 229, 241, 242, 248, 249, 252, 254, 255, 256, 257, 258, 261, 266, 273, 281, 284, 286, 287, 294, 295, 298, 306, 313, 316, 318, 319, 320, 321, 322, 341, 347, 363, 365, 396, 398, 399, 403, 405, 429, 437, 454, 455, 457, 458, 483, 484, 485, 497, 498, 504, 505, 508, 510, 511}.

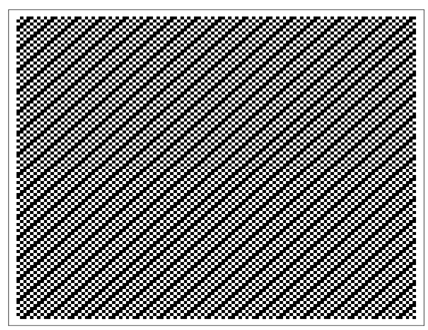

Fig. 2: The evolution of rule 41 from an initial configuration of Λ_1.

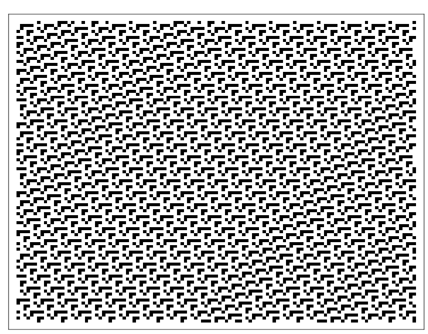

Fig. 3: The evolution of rule 41 from an initial configuration of Λ_2.

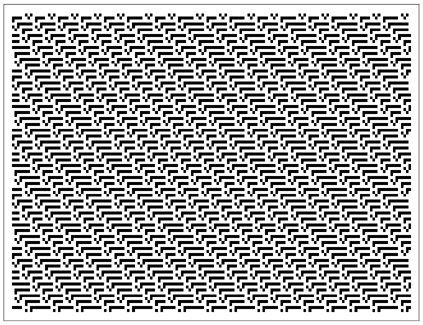

Fig. 4: The evolution of rule 41 from an initial configuration of Λ_3.

Thus, Λ_1, Λ_2 and Λ_3 are subshifts of finite type of f_{41}. For illustration, simulations on Λ_1, Λ_2 and Λ_3 are shown in Figure 2, 3 and 4.

4. Complex Dynamics

In this section, the complexity and chaotic dynamics of f_{41} are explored. Since the topological dynamics of a subshift of finite type is largely determined by the properties of its transition matrix, it is helpful to briefly review some definitions about irreducible and aperiodic [18]. A matrix A is positive if all of its entries are non-negative, irreducible if $\forall\ i, j$, there exists n such that $A^n_{ij} > 0$, aperiodic if there exists N, such that $A^n_{ij} > 0$, $n > N$, $\forall i, j$. If $\Lambda_{\mathscr{A}}$ is a two-order subshift, then it is topologically mixing if and only if A is aperiodic, topological transitive if and only if A is irreducible. A is the associated transition matrix of subshift with $A_{ij} = 1$, if $(i, j) \prec \mathscr{A}$; otherwise $A_{ij} = 0$.

The topologically conjugate relationship between $(\Lambda_{\mathscr{A}}, \sigma)$ and a two-order subshift of finite type can be established, and the dynamical behavior of f_{41} on $\Lambda_{\mathscr{A}}$ can be discussed based on existing results.

Let $\hat{S} = \{s_0, s_1, s_2, s_3\}$ be a new symbolic set, where s_i $(i = 0, 1, 2, 3)$ stand for (0, 1, 0), (0, 1, 1), (1, 0, 1) and (1, 1, 0) appeared in Proposition 3, then one can construct a new symbolic space $\hat{S}^Z$ on $\hat{S}$. Let $\hat{\mathscr{A}} = \{(s, s')|\ s = (b_1, b_2, b_3),\ s' = (b'_1, b'_2, b'_3) \in \mathscr{B}_1, b_j = b'_{j-1}, 2 \le j \le 3\}$, where $\mathscr{B}_1$ is the determinative block system of Λ_1 in Proposition 3. Further, the 2-order subshift $\Lambda_{\hat{\mathscr{A}}}$ of σ is defined by $\Lambda_{\hat{\mathscr{A}}} = \{(\dots, r_{-1}, r_0, r_1, \dots) \in \hat{S}^Z|\ r_i \in \hat{S}, (r_i, r_{i+1}) \prec \hat{\mathscr{A}}, \forall\ i \in Z)\}$. Thus, the transition matrix A_1 of the subshift $\Lambda_{\hat{\mathscr{A}}}$ is

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition 6: $(\Lambda_{\mathscr{B}_1}, \sigma)$ and $(\Lambda_{\hat{\mathscr{A}}}, \sigma)$ are topologically conjugate; namely, $(\Lambda_{\mathscr{B}_1}, f_{41})$ and $(\Lambda_{\hat{\mathscr{A}}}, \sigma)$ are topologically conjugate.

Proof: Define a map from $\Lambda_{\mathscr{B}_1}$ to $\Lambda_{\hat{\mathscr{A}}}$ as follows :

$$g : \Lambda_{\mathscr{B}_1} \to \Lambda_{\hat{\mathscr{A}}}$$

$$(\dots, x_{-1}, \overset{*}{x}_0, x_1, \dots) \to (\dots, r_{-1}, \overset{*}{r}_0, r_1, \dots)$$

where $r_i = (x_i, x_{i+1}, x_{i+2}) \in \mathscr{B}_1$, $\forall\ i \in Z$. One can easily check that g is a homeomorphism and $g \circ \sigma = \sigma \circ g$. Therefore, $(\Lambda_{\mathscr{B}_1}, \sigma)$ and $(\Lambda_{\hat{\mathscr{A}}}, \sigma)$ are topologically conjugate. ∎

It is known that σ is topological mixing on $\hat{\mathscr{A}}$ if and only if the transition matrix A_1 is aperiodic. Thus, the following theorem is obtained.

Theorem 3:
(1) $f_{41}|_{\Lambda_1} = \sigma_L|_{\Lambda_1}$ is topological mixing;
(2) The topological entropy of $f_{41}|_{\Lambda_1}$ satisfies $ent(f_{41}|_{\Lambda_1}) = log(\rho(A_1)) \approx 0.1221$, where $\rho(A_1)$ is the spectral radius of the transition matrix A_1.

A positive topological entropy is often taken as a signature of chaos. A system with topological mixing property has many chaotic properties in different senses such as Devaney and Li-Yorke.

Corollary 1:
(1) $f_{41}|_{\Lambda_1}$ is chaotic in the sense of Li-Yorke;
(2) $f_{41}|_{\Lambda_1}$ is chaotic on Λ_1 in the sense of Devaney.

Theorem 4:
(1) $f_{41}|_{\Lambda_2}$ is topological transitive on Λ_2;
(2) $f_{41}|_{\Lambda_2}$ is chaotic in the sense of Li-Yorke on Λ_2.

Proof: (1) It can be proved that the transition matrix A_2 corresponding to the subshift Λ_2 is irreducible, so $\sigma_L|_{\Lambda_2}$ is topological transitive. Thus, $\sigma^3_L|_{\Lambda_2}$ is topological transitive. It follows from proposition 4 that $f^4_{41}|_{\Lambda_2} = \sigma^3|_{\Lambda_2}$, so $f^4_{41}|_{\Lambda_2}$ is topological transitive. To prove $f_{41}|_{\Lambda_2}$ is topological transitive, one only need to check that for any two open sets $U,\ V \subset \Lambda_2, \exists N > 0$, such that $f^N_{41}(U) \bigcap V \neq \phi$, since $f^4_{41}|_{\Lambda_2}$ is topological transitive, thus, $\exists\ k > 0$, such that $(f^4_{41})^k(U) \bigcap V \neq \phi$. Hence $\exists N = 4k$, such that $f^N_{41}(U) \bigcap V \neq \phi$, this means $f_{41}|_{\Lambda_2}$ is topological transitive.

(2) It follows that $ent(\sigma^3_L|_{\Lambda_2}) = 3 \cdot ent(\sigma_L|_{\Lambda_2})$. It is easy to compute that $ent(f_{41}|_{\Lambda_2}) = \frac{ent(f^4_{41}|_{\Lambda_2})}{4} = \frac{3 \times log(\rho(A_2))}{4} \approx 0.10657$. ∎

Similarly, the dynamical behavior of f_{41} on Λ_3 can be analyzed via the property of the corresponding transition matrix. But the transition matrix A_3 is neither irreducible nor aperiodic. In order to investigate the dynamic behavior on the subshift of finite type Λ_3. Now, take two invariant sets Λ^1_3, $\Lambda^2_3 \subset \Lambda_3$, their determinative block systems are 9-sequence sets $\mathscr{B}^1_3$ and $\mathscr{B}^2_3$ respectively. Let their decimal codes are $D(\mathscr{B}^1_3)$={21, 42, 85, 101, 170, 202, 229, 298, 341, 405, 458, 485} and $D(\mathscr{B}^2_3)$={0, 1, 2, 4, 5, 8, 10, 16, 20, 32, 33, 34, 40, 50, 57, 60, 62, 63, 64, 65, 66, 68, 76, 78, 79, 80, 100, 114, 120, 121, 124, 126, 127, 128, 129, 130, 133, 136, 153, 156, 158, 159, 160, 161, 182, 201, 214, 218, 228, 241, 242, 248, 249, 252, 254, 255, 256, 257, 258, 261, 266, 273, 281, 284, 286, 287, 294, 295, 306, 313, 316, 318, 319, 320, 321, 322, 347, 363, 365, 396, 398, 399, 403, 429, 437, 454, 455, 457, 483, 484, 497, 498, 504, 505, 508, 510, 511}. Moreover, the topological entropy of $f_{41}|_{\Lambda_3^2}$ is $ent(f_{41}|_{\Lambda_3^2}) = log(\rho(A_3^2)) \approx 0.294$, where A_3^2 is the transition matrix corresponding to Λ_3^2.

Theorem 5: $f_{41}|_{\Lambda_3^2}$ is chaotic in the sense of Li-Yorke.

It is obviously that the subshift of finite type $\Lambda_1, \Lambda_2, \Lambda_3 \subset \Lambda$. The dynamical behavior of rule 41 are very complex on its subshift attractor. However, the dynamics should be explored on the invariant set Λ. It has been proved that the only subshift attractor of a surjective cellular automata is the full space [19]. Additive one-dimensional cellular automata defined on a finite alphabet of prime cardinality are chaotic in the sense of Devaney [22]. Rule 41 is not a surjective CA, since it is useful to study the dynamical behavior and to find its maximum attractor. In spite of the dynamics of f_{41} on the finite type subshifts has been made clear, but the ones on Λ should be further studied.

5. Conclusion

One of the main challenges is to explore the quantitative dynamics in cellular automata evolution [24]. this work has developed an elementary and rigorous proof to predict the rich and complex dynamics of rule 41 in view of symbolic dynamical systems. For example, the rule is topological mixing, topological transitive, and possesses positive topological entropies on some subshifts of finite type, thus, it is chaos in the sense of Li-Yorke or Devaney. At the same time, an invariant set is found, which includes the maximum attractor of rule 41. Indeed, the dynamics of rule 41 have not been completely revealed. Some new methods should be exploited to investigate the subshift in future study.

Acknowledgments

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